Calogero-Sutherland model with twisted boundary condition

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In this work, the Calogero-Sutherland model with twisted boundary condition is studied. The ground state wave functions, the ground state energies, and the full energy spectrum are provided in details. [S1063-651X(96)01710-2]

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Exact solutions have provided us with important nonperturbative insights in dealing with systems of strong correlations. While there are very few exactly solvable systems available, the ones that exist have yielded many interesting results. Notable examples include electron systems with δ function interactions [1], the Hubbard model [2], the Kondo impurity spin system with linear conduction electron system [3], the Luttinger model [4] , and the Anderson model [5]. These models have all played important roles in our understanding of physics in condensed matter theory. Ever since Haldane and Shastry independently introduced the exactly solvable spin chain of $1/r^2$ exchange interaction [6,7], there has been considerable activity in studying the variants of the Haldane-Shastry spin chain doped with holes i.e., the t-J models of long range hopping and exchange [8,9,11–16,18–23] It is interesting that the chiral Hubbard model [10], which at half-filling and in the limit of large but finite on-site energy reduces to the Haldane-Shastry spin chain, is also exactly solvable for any filling numbers and any on-site energy. In the following, we will study in details the Calogero-Sutherland (CS) model with twisted boundary condition. The ground state wave functions, the ground state energies, and the full energy spectrum are provided. Since one has to deal with the cases of bosons and fermions with twisted boundary condition, the full discussion is divided into several sections as below.

I. THE GROUND STATES

A. Spinless boson gas

We first consider the CS model of boson gas defined on a closed ring of length L. In the presence of a flux tube that threads through the ring, the eigenenergy problem can be formulated as follows. Suppose that there are N spinless bosons moving on the ring, $0\leqslant x_i\leqslant L; i=1,2,\ldots,N$. Then the eigenvalue problem is

$$H_{CS}\tilde{\Psi}(x_1\sigma_1,\ldots,x_i\sigma_i,\ldots,x_N\sigma_N)=E\tilde{\Psi}(x_1\sigma_1,\ldots,x_i\sigma_i,\ldots,x_N\sigma_N). \quad (1)$$

Here the Calogero-Sutherland Hamiltonian H_{CS} takes the usual form

$$H_{CS}=-\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}+\sum_{i<j} l(l+1)\Bigg/\left[\left(\frac{L}{\pi}\right)^2\sin^2\left(\frac{\pi(x_i-x_j)}{L}\right)\right], \quad (2)$$

where we assume $l>0$. The wave function obeys the twisted boundary condition

$$\tilde{\Psi}(x_1,\ldots,(x_i+L),\ldots,x_N)=e^{i\phi}\tilde{\Psi}(x_1,\ldots,x_i,\ldots,x_N). \quad (3)$$

Obviously, the system is invariant under the translational operation $\phi\rightarrow\phi+2\pi$. Therefore we only need to consider the region $-\pi\leqslant\phi\leqslant\pi$. For the bosons,

$$\tilde{\Psi}(x_1,\ldots,x_i,\ldots,x_j,\ldots)=\tilde{\Psi}(x_1,\ldots,x_j,\ldots,x_i,\ldots,x_N), \quad (4)$$

i.e., the wave function $\tilde{\Psi}$ is symmetric under exchange of any two particles.

Let us define the region R as follows: $\{R:0\leqslant x_i\leqslant L; i=1,2,\ldots,N\}$. The subregion of the full R is denoted by $R_1:\{R_1:0\leqslant x_1\leqslant x_2\leqslant\cdots\leqslant x_N\leqslant L\}$. The wave function inside the region R_1 is denoted by $\tilde{\Psi}_1(x_1,x_2,\ldots,x_N)$. The wave function in other subregions can be obtained by using the symmetry property of bosons Eq. (4). The twisted boundary condition Eq. (3) is translated to be

$$\tilde{\Psi}_1(x_2,x_3,\ldots,x_N,L)=e^{i\phi}\tilde{\Psi}_1(0,x_2,x_3,\ldots,x_N). \quad (5)$$

The ground state of the spinless bosons should take the following form:

$$\tilde{\Psi}_1^g(x_1,x_2,\ldots,x_N)=\exp\left(\frac{i\phi}{L}\sum_{j=1}^{N}x_j\right)\prod_{1\leqslant i<j\leqslant N}\left|\sin\left(\frac{\pi(x_i-x_j)}{L}\right)\right|^{l+1}. \quad (6)$$

One may check that the wave function satisfies the twisted boundary condition. This wave function is also an eigenstate of the Hamiltonian. Since the wave function has no zeros in the region R_1, it is the ground state. The eigenenergy of the state can be found to be

$$E_g(\phi)=\frac{1}{2}N\left(\frac{\phi}{L}\right)^2+\frac{1}{6}(l+1)^2\pi^2N(N^2-1)/L^2. \quad (7)$$

Without the flux, the results reduce to those of Sutherland's [17]. In the full space R, the ground state wave function $\tilde{\Psi}^g$ takes the simple form

$$\tilde{\Psi}^g(x_1,x_2,\ldots,x_N)=\exp\left(\frac{i\phi}{L}\sum_{j=1}^{N}x_j\right)\prod_{1\leqslant i<j\leqslant N}\left|\sin\left(\frac{\pi(x_i-x_j)}{L}\right)\right|^{l+1}, \quad (8)$$

which is symmetric under the exchange of two particles, and which satisfies the twisted boundary condition Eq. (3). In the presence of the flux, there is a persistent current in the ring. The persistent current is $I(\phi)=-[\partial E_g(\phi)/\partial\phi]=-(N/L^2)\phi$.

B. Spinless fermion gas (odd N)

In this section, we discuss the spinless fermion model described by the CS model in the presence of a magnetic flux tube. The eigenvalue problem is formulated as follows:

$$H_{CS}\tilde{\Psi}(x_1\sigma_1,\ldots,x_i\sigma_i,\ldots,x_N\sigma_N)=E\tilde{\Psi}(x_1\sigma_1,\ldots,x_i\sigma_i,\ldots,x_N\sigma_N), \quad (9)$$

with the Calogero-Sutherland Hamiltonian H_{CS} as before

$$H_{CS}=-\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}+\sum_{i<j}l(l+1)\Bigg/\left[\left(\frac{L}{\pi}\right)^2\sin^2\left(\frac{\pi(x_i-x_j)}{L}\right)\right], \quad (10)$$

where the coupling constant $l>0$. The wave function satisfies the twisted boundary condition

$$\tilde{\Psi}(x_1,\ldots,(x_i+L),\ldots,x_N)=e^{i\phi}\tilde{\Psi}(x_1,\ldots,x_i,\ldots,x_N). \quad (11)$$

In this case, since the system is made of spinless fermions, the wave function is antisymmetric when exchanging two particles,

$$\tilde{\Psi}(x_1,\ldots,x_i,\ldots,x_j,\ldots)=(-1)\tilde{\Psi}(x_1,\ldots,x_j,\ldots,x_i,\ldots,x_N). \quad (12)$$

As before, we define the full region R to be $\{R:0\leqslant x_i\leqslant L;i=1,2,\ldots,N\}$. The subregion R_1 is $\{R_1:0\leqslant x_1\leqslant x_2\leqslant\cdots\leqslant x_N\leqslant L\}$. The twisted boundary condition of the wave function $\tilde{\Psi}(x_1,x_2,\ldots,x_N)$ defined in R is translated to a condition satisfied by the wave function $\tilde{\Psi}_1(x_1,x_2,\ldots,x_N)$ defined in the region R_1 as below

$$\tilde{\Psi}_1(x_2,x_3,\ldots,x_N,L)=(-1)^{(N-1)}e^{i\phi}\tilde{\Psi}_1(0,x_2,x_3,\ldots,x_N). \quad (13)$$

Given $\tilde{\Psi}_1$ inside R_1, the wave functions in other subregions of R can be obtained using the antisymmetry of the fermionic statistics. Since N is odd, the prefactor $(-1)^{(N-1)}$ disappears. We propose the following wave function as the ground state: inside R_1, the ground state takes the Jastrow form

$$\tilde{\Psi}_1^g(x_1,x_2,\ldots,x_N)=\exp\left(\frac{i\phi}{L}\sum_{j=1}^{N}x_j\right)\prod_{1\leqslant i<j\leqslant N}\left|\sin\left(\frac{\pi(x_i-x_j)}{L}\right)\right|^{l+1}. \quad (14)$$

One may compute the corresponding eigenvalue of this wave function. It is found that

$$E_g(\phi)=\frac{1}{2}N\left(\frac{\phi}{L}\right)^2+\frac{1}{6}(l+1)^2\pi^2N(N^2-1)/L^2. \quad (15)$$

In the presence of the magnetic flux, there is a persistent current in the system. The persistent current is $I(\phi)=-[\partial E_g(\phi)/\partial\phi]=-(N/L^2)\phi$. Without the flux, our ground state wave function reduces to that of Sutherland's. In the full region R, the ground state wave function $\tilde{\Psi}^g$ can be written in a compact way

$$\tilde{\Psi}^g(x_1,x_2,\ldots,x_N)=\exp\left(\frac{i\phi}{L}\sum_{j=1}^{N}x_j\right)\prod_{1\leqslant i<j\leqslant N}\times\left|\sin\left(\frac{\pi(x_i-x_j)}{L}\right)\right|^{l}\sin\left(\frac{\pi(x_i-x_j)}{L}\right), \quad (16)$$

which is antisymmetric when exchanging two fermions, and which also satisfies the twisted boundary condition Eq. (13).

C. Spinless fermion gas (even N)

The eigenvalue problem is formulated as before. However, great care should be taken of the boundary condition Eq. (13). First, let us consider the situation without flux, $\phi=0$, and we impose the periodic boundary condition (PBC) on the wave function

$$\tilde{\Psi}(x_1,\ldots,(x_i+L),\ldots,x_N)=\tilde{\Psi}(x_1,\ldots,x_i,\ldots,x_N). \quad (17)$$

This PBC is translated to be a boundary condition for $\tilde{\Psi}_1$ as below

$$\tilde{\Psi}_1(x_2,x_3,\ldots,x_N,L)=(-1)\tilde{\Psi}_1(0,x_2,x_3,\ldots,x_N). \quad (18)$$

With this in mind, the ground state for the system inside the region R_1 should take the following form:

$$\tilde{\Psi}_1^g(x_1,x_2,\ldots,\quad x_N)=\exp\left(\frac{\pm i\pi}{L}\sum_{j=1}^N x_j\right)\times\prod_{1\leq i<j\leq N}\left|\sin\left(\frac{\pi(x_i-x_j)}{L}\right)\right|^{l+1}. \tag{19}$$

We can compute the eigenenergy of this wave function. The ground state energy is found to be

$$E_g=\frac{1}{2}N\left(\frac{\pi}{L}\right)^2+\frac{1}{6}(l+1)^2\pi^2N(N^2-1)/L^2. \tag{20}$$

This energy is different from the bosonic case, as well as different from the fermionic case when the total number of particles is odd.

Now, let us consider the situation when there is nonzero flux. Consider the case where $0\leq\phi\leq\pi$. The twisted boundary condition Eq. (13) for the wave function $\tilde{\Psi}_1$ is satisfied by the following Jastrow product:

$$\tilde{\Psi}_1^g(x_1,x_2,\ldots,x_N)=\exp\left(\frac{i(\phi-\pi)}{L}\sum_{j=1}^N x_j\right)\times\prod_{1\leq i<j\leq N}\left|\sin\left(\frac{\pi(x_i-x_j)}{L}\right)\right|^{l+1}. \tag{21}$$

This wave function is the ground state of the system, with the ground state energy given by

$$E_g(\phi)=\frac{1}{2}N\left(\frac{\pi-\phi}{L}\right)^2+\frac{1}{6}(l+1)^2\pi^2N(N^2-1)/L^2. \tag{22}$$

The persistent current of the system is found to be $I(\phi)=-\partial E(\phi)\partial\phi=-N/L^2(\phi-\pi)\geq 0$. If the system has a flux $-\pi\leq\phi\leq 0$, the ground state wave function is found to be

$$\tilde{\Psi}_1^g(x_1,x_2,\ldots,x_N)=\exp\left(\frac{i(\phi+\pi)}{L}\sum_{j=1}^N x_j\right)\times\prod_{1\leq i<j\leq N}\left|\sin\left(\frac{\pi(x_i-x_j)}{L}\right)\right|^{l+1}. \tag{23}$$

The corresponding eigenenergy is given by

$$E_g(\phi)=\frac{1}{2}N\left(\frac{\pi+\phi}{L}\right)^2+\frac{1}{6}(l+1)^2\pi^2N(N^2-1)/L^2. \tag{24}$$

The persistent current is $I(\phi)=-N/L^2(\phi+\pi)\leq 0$.

In the full space R as defined before, the ground state wave function $\tilde{\Psi}^g$ can be written in a compact way. For $0\leq\phi\leq\pi$, inside the full region R, the ground state wave function is

$$\tilde{\Psi}^g(x_1,x_2,\ldots,x_N)=\exp\left(\frac{i(\phi-\pi)}{L}\sum_{j=1}^N x_j\right)\times\prod_{1\leq i<j\leq N}\left|\sin\left(\frac{\pi(x_i-x_j)}{L}\right)\right|^{l}\times\sin\left(\frac{\pi(x_i-x_j)}{L}\right). \tag{25}$$

While for $-\pi\leq\phi\leq 0$, inside R, one has

$$\tilde{\Psi}^g(x_1,x_2,\ldots,x_N)=\exp\left(\frac{i(\phi+\pi)}{L}\sum_{j=1}^N x_j\right)\prod_{1\leq i<j\leq N}\left|\sin\left(\frac{\pi(x_i-x_j)}{L}\right)\right|^{l}\times\sin\left(\frac{\pi(x_i-x_j)}{L}\right). \tag{26}$$

In the next section, we will provide the full energy spectrum for the CS model under twisted boundary condition, following a similar approach of Sutherland's for zero flux case [17].

II. EXCITATION SPECTRUM

A. Spinless boson gas

Following the idea of Sutherland [17], one can write the wave function as a product of the Jastrow part and the part of plane waves. Keeping the plane waves due to the twisted boundary condition we can find the energy spectrum of the spinless boson gas given by

$$E=\frac{\pi^2}{6}(l+1)^2N(N^2-1)/L^2+\frac{1}{2}(2\pi/L)^2\epsilon, \tag{27}$$

where the function ϵ is given by $\epsilon=\Sigma_{j=1}^N(n_j+[\phi/2\pi])^2+(l+1)\Sigma_{i>j}[n_i-n_j]$. The quantum numbers n_j are nonnegative integers, which satisfy the condition $n_{j+1}\geq n_j$. The quantum numbers do not have to be distinct from each other. The ground state is obtained when all $n_j=0$.

B. Spinless fermions (odd N)

For the spinless fermion gas (odd N), one can also find the excitation spectrum of the system under the twisted boundary condition. The full energy spectrum takes the form

$$E=\frac{\pi^2}{6}(l+1)^2N(N^2-1)/L^2+\frac{1}{2}(2\pi/L)^2\epsilon, \tag{28}$$

where the function ϵ is given by $\epsilon=\Sigma_{j=1}^N(n_j+[\phi/2\pi])^2+(l+1)\Sigma_{i>j}[n_i-n_j]$. The quantum numbers n_j are non-negative integers, and one has the condition $n_{j+1}\geq n_j$. The ground state is given when all $n_i=0$.

C. Spinless fermions (even N)

Finally, for the spinless fermion gas of even N, we also find the full energy spectrum taking the following form for $0\leq\phi\leq\pi$:

$$E=\frac{\pi^2}{6}(l+1)^2N(N^2-1)/L^2+\frac{1}{2}(2\pi/L)^2\epsilon, \qquad (29)$$

where the function ϵ is given by $\epsilon=\Sigma_{j=1}^{N}(n_j+[(\phi-\pi)/2\pi])^2+(l+1)\Sigma_{i>j}[n_i-n_j]$. The quantum numbers n_j are non-negative integers that have the condition $n_{j+1}\geqslant n_j$. The ground state corresponds to all $n_i=0$.

When the flux $-\pi\leqslant\phi\leqslant 0$, the energy excitation is given by

$$E=\frac{\pi^2}{6}(l+1)^2N(N^2-1)/L^2+\frac{1}{2}(2\pi/L)^2\epsilon, \qquad (30)$$

where the function ϵ is given by $\epsilon=\Sigma_{j=1}^{N}(n_j+[(\phi+\pi)/2\pi])^2+(l+1)\Sigma_{i>j}[n_i-n_j]$. The quantum numbers n_j are non-negative integers, satisfying the condition $n_{j+1}\geqslant n_j$. The ground state is reached when all $n_i=0$.

III. SUMMARY

In summary, we have discussed how the boundary condition affects the spinless CS model of long range interaction. The ground state wave functions, the ground state energies, the full energy spectrum are provided for both the fermionic gas and the bosonic gas. The exact solutions indicate that the parity effect for the persistent currents still hold for the fermionic gas, in spite of the electron-electron correlation.

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